Nader El-Bizri

THE MATHEMATICAL ORDERS OF ARCHITECTURE: SEEING MADĪNAT AL-ZAHRĀʾ FROM THE PERSPECTIVE OF THE IKHWĀN AL-ṢAFĀʾ

Introduction

This study focuses on the applications of mathematics in the analysis of selected architectural features of Madīnat al-Zahrāʾ (Radiant City; Ville radieuse; fourth/tenth century) in the outskirts of Córdoba, with an emphasis on the ordering principles behind the design of the Salón Rico/Hall of Richness of the Umayyad caliph ʿAbd al-Raḥmān III (r. 299-350/912-61). To mediate this analysis via the mathematical knowledge of that epoch in the Islamicate milieu, I evoke some principal guiding leitmotifs from the epistles on arithmetic, geometry, and proportional ratios (*nisab*) of the Ikhwān al-Ṣafāʾ (Brethren of Purity), as these were set in their *Rasāʾil* (epistles) 1, 2, and 6 respectively. I also consider the applications of these mathematical subfields to the arts and crafts of construction techniques as embodied in the *Kitāb fī mā yaḥtāj ilayhi al-ṣāniʿ min aʿmāl al-handasa* and *Kitāb fī ʿamal al-misṭara wa-l-birkār wa-l-kūnyā* composed by Abū l-Wafāʾ al-Būzjānī (d. 388/998)[1].

Madīnat al-Zahrāʾ

In terms of a broad perspective on the locale and architectural setting analyzed here, Madīnat al-Zahrāʾ is a fortified palatine urban

1. I projected and constructed all the geometric figures included in the present article to illustrate the mathematical underpinnings of the architectural elements based on the classical technical sources from the Islamicate milieu that are evoked in the body of the text.

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complex in the western environs of Córdoba, situated at the foothills of Jabal al-ʿArus. It lasted as a settlement from 324/936 to 404/1013, when it was sacked and burnt by Berber troops. It contained ceremonial reception halls, a mosque and prayer spaces, government offices, gardens, a minting workshop, the Dār al-Jund (lit., soldiers' house) military barracks with its equestrian quarters, in addition to various residences and bathhouses. The construction materials included polychromatic stones, iron, marble, timber, and gold accents. It was organized in three large terraces surrounded by an external boundary wall. The upper locale was dedicated to the palace, the middle grounds contained the bureaucratic halls, the military barracks, equestrian quarters, and gardens, while the lower plateau was meant to accommodate the residences of commoners, and also included a souk and a mosque.

The Hall of Richness

The "Hall of Richness" (Salón Rico) of ʿAbd al-Raḥmān III was located in the central part of Madīnat al-Zahrāʾ, and served as one of the principal ceremonial reception spaces of the eastern assembly (*al-majlis al-sharqī*), which was also known as "the convivial" (*al-muʾnis*). Its design has a hypostyle shape, with a vestibule and a tripartite space; its naves were delimited by rows of double-tiered horseshoe arches, which also featured a central blind arch that demarcated a niche to house the seat of the caliph.

Epigraphic inscriptions were carved on the bases and pilasters of the structural elements of that space, and these carried the signatures of chief masons: Bard, Naṣr, Fattāh, Aflaḥ, Ṭāriq, as supervised by the master architect, Maslama b. ʿAbdallāh, and assisted by other builders and artisans from Baghdad, Damascus, Constantinople, and Carthage [2].

2. See, for example, A. Almagro, «Análisis Tipológico de la Arquitectura Residencial de Madinat al-Zahra», in *Al-Andalus und Europa: Zwischen Orient und Okzident*, ed. M. Müller-Wiener, Ch. Kothe, K.-H. Golzio, J. Gierlichs (Petersberg 2004), 117-24; M. A. Almansa, «Materiales e hipótesis para una interpretación del Salón de Abd al-Rahman al-Nasir», in *Madinat al-Zahara: El Salón de Abd al-Rahman III*, ed. A. Vallejo Triano (Córdoba 1995), 177-95; A. Vallejo Triano (ed.), *Madinat al-Zahra: El Salón de Abd al-Rahman III* (Córdoba 1995).

This initial description gives us a broad setting in which we can consider the architectural and surveying analytics of this site; this is especially true of the Hall of Richness[3].

Recent measurements were conducted by modern architects, archeologists, and surveyors, via geometrical matrices, and projected onto the Hall of Richness space to determine its design elements. In this regard, we should mention the studies of Francisco Javier Roldán-Medina who developed a geometric method of mensuration to scale premodern architectural edifices in general, and the inquiries of Felix Arnold, who studied the ordering principles of façades and the elevations of spatial volumes along with plan views. For example, Roldán-Medina used the base duodecimal arithmetic and geometrical constructions that are extracted from the diagonal of a square to analyze the double metric scale in the modulation of the sizes and proportions of the architectural structures[4]. Arnold disclosed the mathematical aspects that underpin the architectural structure of equilateral triangles for floor plan metrics, in addition to the squares and diagonals as and in measurements. He also showed that these metrics were applied to arcades, with a particular focus on the army barracks of Dār al-Jund, how the horseshoe arch reduced the height and width of the supporting columns in proportion to the scaling of the elevation, and how such parameters unified the volumetric space and not only its surfaces[5].

The layout of the Hall of Richness is rectangular in its plan view, with the interior arcades shaped in terms of a sequence of load-bearing horseshoe arches. Its external elevation features a five-bay arcade of a tessellation of decorative horseshoe arches on the façade.

3. Various sources give historical details and topographical descriptions as well as architectural particulars regarding Madīnat al-Zahrā'. See I. Arias, L. Balmaseda, Á. Franco, C. Papí, «Documentación, Inventario y Catalogación de los Materiales Procedentes de Medina Azahara (Córdoba) en el Museo Arqueológico Nacional», in *Boletín del Museo Arqueológico Nacional* 19 (2001), 88-127; D. F. Ruggles, *Madinat al-Zahra', Gardens, Landscape, and Vision in the Palaces of Islamic Spain* (University Park, PA 2000), 53-85; A. Vallejo Triano, «Madinat al-Zahra: The Triumph of the Islamic State», in *al-Andalus: The Art of Islamic Spain*, ed. J. D. Dodds (New York 1992), 27-39.

4. F. J. Roldán-Medina, «Method of Modulation and Sizing of Historic Architecture», in *Nexus Network Journal* 14 (2012): 539-53.

5. F. H. Arnold, «Mathematics and the Islamic Architecture of Córdoba», in *Arts: Multidisciplinary Digital Publishing Institute* 7 (2018): 1-15.

The interlaced patterns of ornamentation on these arcades were executed in marble for the columns, capitals, pavements, and revetments, while the wall engravings were accented by sculpted stucco (see figure 1).

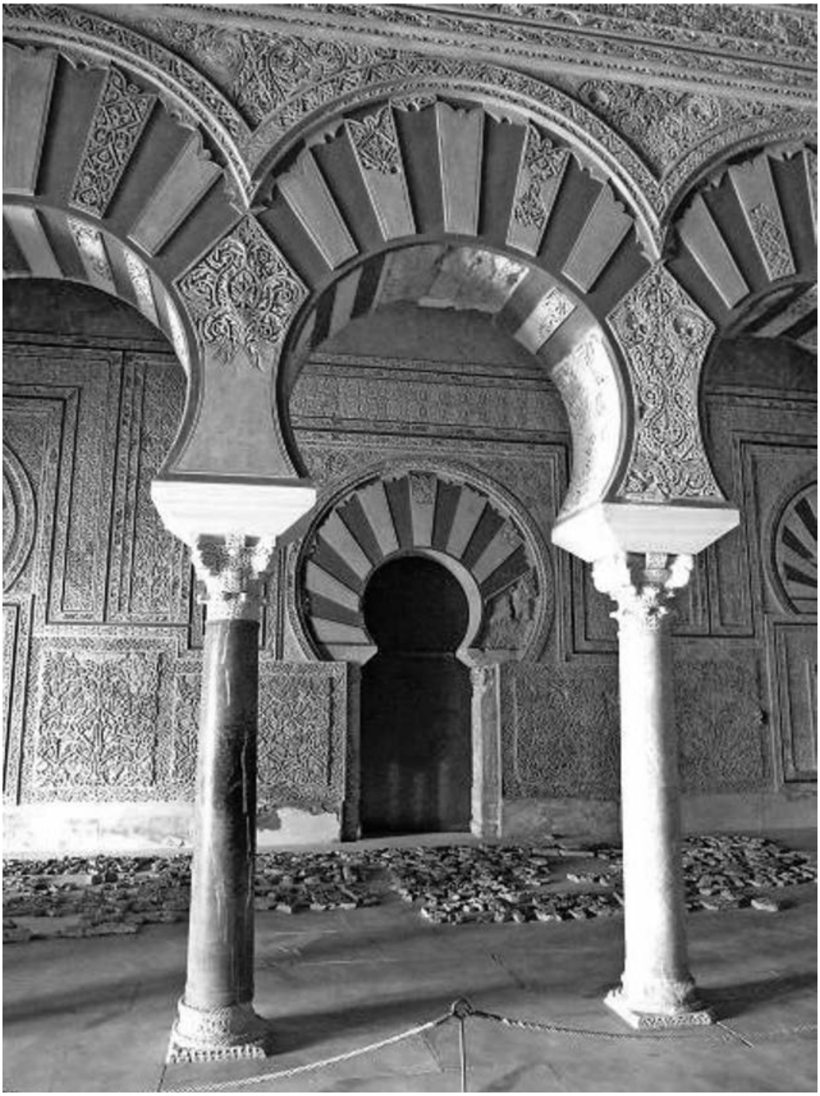

Fig. 1. Hall of Richness horseshoe arch. Image obtained through open access photography from Wikimedia Commons, February 2023.

Horseshoe Arch

Considering the case of the horseshoe arch as a structural *voussoir*, its architectural shape is determined by the mechanics of its load-bearing curvature, which carries the weight of the upper floor and conveys the stresses downward, stopping at its flanking alcoves. This describes a masonry technique of building a load-bearing arch without the need for a keystone. Similar structural features were used in the double-tiered horseshoe arches of the mosque of Córdoba; they also appear in the Baptistery of St. Jacob of Nisibis (Mār Yaʿqūb) in upper Mesopotamia, and as a rudimentary form in the Iberian Visigoth Santa Eulalia de Bóveda de Mera of Lugo[6]. Figure 2 shows the

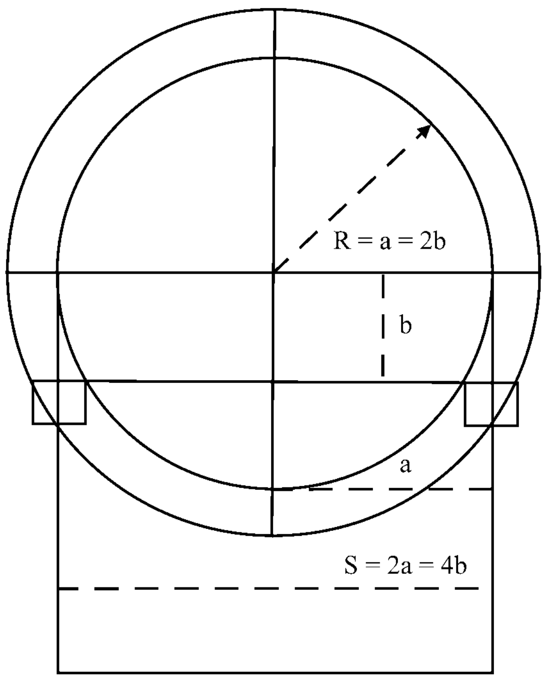

Fig. 2. Load-bearing structure of the horseshoe arch.

6. Enrique Jorge Montenegro Rúa et al., *Santa Eulalia de Bóveda* (Santiago de Compostela: Consellería de Cultura, Educación e Ordenación Universitaria, Drirección Xeral do Patrimonio Cultural, 2008).

generic geometric principles that determine the construction of the horseshoe arch and the mechanics of its load-bearing structural properties.

If the radius R of this horseshoe arch (as shown in figure 2) is equal to a given magnitude *a*, then *a* is half the distance between the respective axes of each of the columns supporting that horseshoe arch, and is therefore equal to *2b*, whereby *b* is the distance between the height of the center of the horseshoe arch and the top of the respective capitals of each of the two columns supporting that horseshoe arch. Following these guiding geometric constructions that inform the mechanics of the building technique and the statics of the curvilinear load distribution and weight bearing shape, we find the horseshoe arch in the Hall of Richness in a form that resembles what I depict in figure 3.

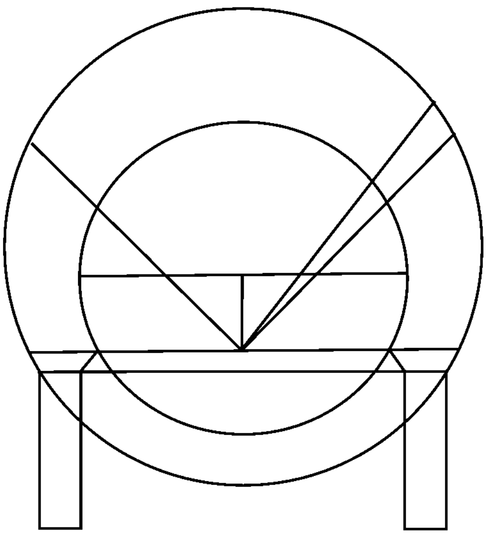

Fig. 3. Geometry of the horseshoe arch.

Mathematical Knowledge

Unlike the attempts to devise an extrinsic geometrical ordering principle through which this Hall of Richness can be surveyed, my aim is to show how the application of mathematical knowledge in its basic forms assists in analyzing the architectural features of this space, the geometric principles that guided the mechanics of its construction, and the structural techniques of building it. This is mediated initially with imprecise measurements to determine the proportionality of the architectural features; I then focus on a mathematical analysis orientated by the epistles on arithmetic, geometry, and proportional ratios (*nisab*) of the Ikhwān al-Ṣafāʾ[7]. I also consider the application of these three mathematical subfields in guiding the construction techniques of the arts and crafts that are accounted for in the *Kitāb fī mā yaḥtāj ilayhi al-ṣāniʿ min aʿmāl al-handasa* (A manual concerning what the maker [builder or craftsman] needs from the works of geometry) and the *Kitāb fī ʿamal al-misṭara wa-l-birkār wa-l-kūnyā* (A manual on using the straightedge ruler, compass, and orthogonal triangular edge) as both were composed by Abū l-Wafāʾ al-Būzjānī[8].

Tracing designs in geometric compositions requires the use of a compass (*birkār*), a straightedge ruler (*misṭara*), and an orthogonal triangular edge (*al-kūnyā*; *al-muthallath al-qāʾim al-zāwiya*; right triangle). Moreover, this necessitates an understanding of trigonometry to set the properties of triangulation. These basic geometric tools and artisanal equipment were combined with on-site construction techniques at the full scale of the architectural feature, and the handling of building materials, which required an understanding of the mechanics of their statics in load-bearing and weight-distribution. The use of straightedge rulers ensured the precise collinear alignments, while the compass (*birkār*) determined equidistance via its traced circularity, and the orthogonal triangular edge (*kūnyā*) secured a perpendicular projection and intersection. In a basic geometric construction, the perpendicular orthogonal projection can be determined by a straight line joining the two points of the intersection between two equal circles; this also set the distance between them as equal to their respective radii. Moreover, in the use of the compass we can trace two circles that are identical with respective equal radii (as shown in figure 4). If these two identical circles

7. *Epistles of the Brethren of Purity. On Arithmetic and Geometry: Epistles 1-2*, ed. N. El-Bizri (Oxford 2012); *Epistles of the Brethren of Purity. On Composition and the Arts: Epistles 6-8*, ed. N. El-Bizri and G. de Callataÿ (Oxford 2018).

8. Abū l-Wafāʾ al-Būzjānī, *Kitāb fī mā yaḥtāj ilayhi al-ṣāniʿ min aʿmāl al-handasa*, ed. S. A. ʿAlī (Baghdad 1979); Abū l-Wafāʾ al-Būzjānī, *Kitāb fī ʿamal al-misṭara wa-l-birkār wa-l-kūnyā*, ed. M. A. Abū Samra (Doha 2019).

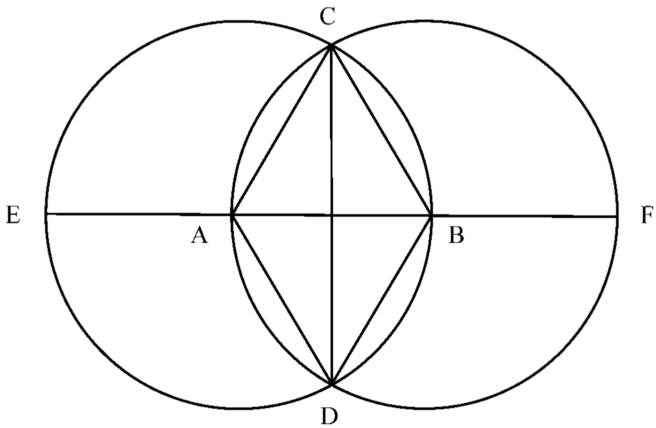

Fig. 4. Geometric construction using a compass, straight edge ruler, and triangular edge.

respectively have centers A and B, and are traced in such a way that their circumferences pass via their centers, then the line AB = μ is equal to the radius of each of these two circles. At the intersection of their circumferences, the two circles have a common chord CD that is perpendicular to their axes as they extend from E to F (CD ⊥ EF). Taking the triangles ACB and ADB, they are also equal in size and they describe two equilateral triangles.

This mode of mathematical knowledge, especially in geometry, and its various applications in generating forms, mainly in architecture and building techniques, was known to al-Būzjānī, and elaborated in his construction manuals. Al-Būzjānī's mathematical knowledge is more basic and practical in the applied arts and crafts of the building techniques than the more theoretical and didactic mathematics of the Ikhwān al-Ṣafā', as set out in the first two epistles on arithmetic and geometry of the *quadrivium* part of the *Rasāʾil*. Their knowledge distills the basics in arithmetic and geometry from the oeuvres of their predecessors (among the mathematicians) and contemporaries. They build on this in specialized treatises on mathematics that stretch back to the times of the Banū Mūsā, Thābit b. Qurra, Ibrāhīm b. Sinān, and included figures like Abū Sahl Wayjan al-Qūhī, who studied conics (*makhrūṭāt*) in the Arabic Archimedean-Apollonian tradition, not simply on the Euclidean-Ptolemaic lineages. For example, al-Qūhī endeavored to study the properties of the perfect geometric compass (*fī l-birkār al-tāmm*) with a view to tracing precise conic sections (namely, the circle [*qaṭʿ dāʾirī*], ellipse [*qaṭʿ nāqiṣ*], parabola [*qaṭʿ mukāfiʾ*], and hyperbola [*qaṭʿ zāʾid*]). This also relates to his studies in mechanics and on the construction of the astrolabe[9].

According to the Ikhwān al-Ṣafāʾ, geometry allowed the precise measurement of magnitudes (*maqādīr*) and distances (*abʿād*) whether in its physical sensible forms as a *handasa ḥissiyya* (as witnessed in architecture), which belonged to the practical crafts (*ṣanāʾiʿ ʿamaliyya*), or in its abstract intelligible modalities under the rubric of the *handasa ʿaqliyya*, which encompassed the theoretical arts (*ṣanāʾiʿ ʿilmiyya*).

The shift from the *handasa ḥissiyya* (sensible geometry) to *handasa ʿaqliyya* (intelligible geometry) corresponds with the abstraction of the forms of sensible entities and positing them as intelligible species (*εἰδος*; *eídos*) that are independent from their material sensory appearances. This is a transition from the sensible to the intelligible (*min al-maḥsūsāt ilā l-maʿqūlāt*), which describes an epistemic process of acquiring abstract knowledge from sensible entities by way of intellection. This fits with the doctrine of hylomorphism; namely, the entanglement of ὕλη (*húlē*) and μορφή (*morphḗ*), as the sensible composition (*tarkīb*) of the matter *hayūlā* (matter) with the *ṣūra* (form) of a given extension (i.e., spatial attributes). The forms of things are abstracted from their material substratum by way of imagination (*takhayyul*; *khayāl*) as it aids the workings of the intellect (*ʿaql*). This discloses the εἶδος (*eidos*) as the essence of a thing *qua* its quiddity (*māhiyya*). This also reflects the method of descriptive geometry in abstracting the form of existing things from their matter, and the process of depicting non-existent new potential entities by way of innovation in design, as for example with architecture or by imagining artworks. Geometry is a systematic process to access eidetic forms in an abstract mode of demonstration that transcends the materiality of sensible entities. This epistemic turn from the sensible to the intelligible underpins the determination of the forms of existing things by way of abstracting their formal fea-

9. J. Sesiano, «Note sur trois théorèmes de mécanique d'al-Quhi et leur conséquence», in *Centaurus* 22 (1979), 281-97; J. L. Berggren, «Abū Sahl al Kūhī's Treatise on the Construction of the Astrolabe with Proof», in *Physis* 31 (1994): 141-252.

tures from their material substrata through geometric imagining and postulation. In another sense, geometric constructions of formal configurations can guide the design by generating a designed shape as a form that is abstracted from matter, and then devising ways of bringing it from potentiality into actuality by an act of fabrication and making. This is the basis of architectural design, and how it becomes a practice of technical building with the crafts of production and construction.

For example, in this context the art of surveying (*misāḥa*) devises geometric procedures for measuring the physical features of existing architectural spaces by generating their formal scaffolding and quantifying them through the metrics of mensuration (in units of feet, arms, palms, fingers, etc.). This is based on an abstract mode of geometric reasoning that shifts from the sensible (*maḥsūs*) to the intelligible (*maʿqūl*), and determines the formal properties of existent entities as hypothetically removed in mathematical imaging from their matter. As for the transition from intellective geometrical knowledge to its sensory mode (*min al-maʿqūlāt ilā l-maḥsūsāt*), it is ultimately the basis for design in generating an architectonic built form or an artwork of the crafts. Architecture (*handasat al-miʿmār*) can be seen in this context as the locus par excellence for this movement between these two modalities of geometric knowledge in its intelligible and sensible manifestations. Similar aspects underpinned a fortiori the operational and procedural usages of geometry in the studies of astronomy, mechanics, and optics.

Ratios and Proportions

Based on the use of arithmetic (*ʿilm al-ʿadad*) and geometry (*ʿilm al-handasa*) in architecture (*handasat al-miʿmār*), the issue of proportionality (*nisab*) becomes fundamental. This is accomplished by assigning numerical values to quantify the scale and measuring of geometric forms. The combination of arithmetic with geometry generates the scaled conditions for designing the architectural form or describing the formal features of existing spaces. The determination of mathematical proportions guides the processes of architectural design and technical execution. The attributes of proportionality are posited in this regard under three types by way of quantity (*kammiyya*), quality (*kayfiyya*), or a combinatorial composition (*murakkab*) of both. The Ikhwān al-Ṣafā᾿ reflect the mathematical knowledge of their age in this regard by highlighting what mathematicians determined as the attributes of proportion and ratio in *Epistle* 6 of their proto-encyclopedic compendium. The mean (*tawassuṭ*) encompass the quantitative *arithmetic proportion* (ἀριθμητικὴ; *arithmetica*; *nisba ʿadadiyya*; *tawassuṭ ʿadadī*), the qualitative *geometrical proportion* (γεωμετρικὴ; *geometrica*; *nisba handasiyya*; *tawassuṭ handasī*), and their composite as a *harmonic musical proportion* (ἁρμονικὴ; *harmonica*; *nisba ta᾿līfiyya*; *tawassuṭ ta᾿līfī*)[10].

To further illustrate these properties, we evoke the following geometric construction (figure 5), that determines the derivation of the arithmetic, geometric, and harmonic means.

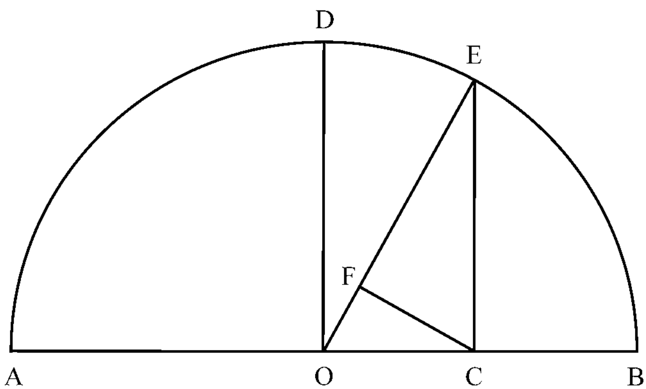

Fig. 5. Construction of the arithmetic, geometric, and harmonic means.

If we have a semicircle with a center O and a diameter AB (as shown in figure 5), and there is a point C at half the distance between O and B, then we project a perpendicular from O to intersect the perimeter of the semicircle at D (OD ⊥ AB), and project a perpendicular from C to intersect the perimeter of the semicircle at E (CE ⊥ AB), and project a perpendicular from C to intersect OE in F (CF ⊥ OE). Based on this construction, we obtain, geometrically, the following determinations of the means: the arithmetic mean *Am* = OD; the geometric mean *Gm* = CE; the harmonic

10. *Epistles, On Composition and the Arts: Epistles 6-8*, 19-21.

mean Hm = EF. Thus, if AC = a, CB = b, and $a > b$, then Am = OD = $\frac{a+b}{2}$; Gm = CE = $\sqrt{ab}$; Hm = EF = $\frac{2ab}{a+b}$.

This describes a principle in geometry that was known to the Ikhwān al-Ṣafāʾ, and reflects their key sources for the theory of ratios and proportions as mediated via the Arabic mathematical adaptive reception of the Ἀριθμητικὴ εἰσαγωγή (*Madkhal ilā ʿilm al-ʿadad*; Introduction to arithmetic; *Introductionis arithmeticae*) of Nicomachus of Gerasa[11], and the Στοιχεῖα (*Kitāb Uqlīdis fī al-uṣūl*; *Elements*) of Euclid[12].

When considering ratios and proportions, the Arabic signifier is encompassed in a single term by the word *nisba*. However, a further implicit clarification is needed to determine the distinction and connection between a ratio (λόγος) and a proportion (ἀναλογία) as both are designated by the single Arabic appellation *nisba*. This notion figures in *Epistle* 6 of the Ikhwān al-Ṣafāʾ, and it builds on what was originally articulated in chapter 21 of Book II of Nicomachus' (*Madkhal ilā ʿilm al-ʿadad*; *Introduction to Arithmetic*; *Introductionis arithmeticae*). Therein, it was stated that a *proportion* [ἀναλογία] is the combination of two or more *ratios* [λόγως]; while a *ratio* is the relation of two terms. The combination of ratios yields a proportion as an equality of two ratios, and a geometric example of this is set when the base of one triangle is twice the base of another triangle, whereby the former has twice the area of the latter with a ratio of 2:1; since both are commensurable by a common measure. Moreover, based on Definition 3 in Book V of Euclid's Στοιχεῖα (*Kitāb Uqlīdis fī l-uṣūl*; *Elements*): "a ratio [λόγος] is the qualitative relation [σχέσις] with reference to size [πηλικότης] between two homologous magnitudes. Proportion [ἀναλογία] is accordingly the likeness of ratios"[13].

The mathematical *milieu* of the predecessors and contemporaries of the Ikhwān al-Ṣafāʾ would have used geometry a fortiori for kinematic constructions and projections. Consequently, they introduced motion into geometry, which has its own ontological entailments

11. Nicomachus of Gerasa, *Introduction to Arithmetic*, trans. M. L. D'Ooge (New York 1926); M. J. Farhān, «Philosophy of Mathematics of Ikhwān al-Ṣafāʾ», in *Journal of Islamic Science* 15 (1999), 25-53.

12. Euclid, *Elements* (*Euclides opera omnia*), ed. J. L. Heiberg and H. Menge (Leipzig 1883-1916).

13. Euclid, *Elements*, Book V, Definitions 3, 6; Euclid, *The Thirteen Books of Euclid's Elements*, trans. Thomas Heath, 2nd ed. (Cambridge 1925).

that go beyond the Greek engagement with mathematics. This is the case given the Platonist bifurcation of the realm of archetypal universal immutable being from the mimetic mutable and apparent particulars of becoming. It is a distinction of the *hyletic* sensible from the *eidetic* intelligible; since motion pertains to the natural *cum* physical realm of the senses in the composition of matter and form, and not to the imaginal domain of intellection in thinking about abstracted forms as separate from matter. Using geometric kinematics, some dynamic rectangles with square roots can be constructed as such by way of projections, and the same applies to the determination of what came to known as the "golden ratio rectangle". Based on these geometric constructions, the dynamic rectangle and the golden ratio can assist in analyzing the horseshoe arch elevation in the Salón Rico; this applies to the study of the design of the arcade of façades and massing volumes of the inner court of the Alhambra as well.

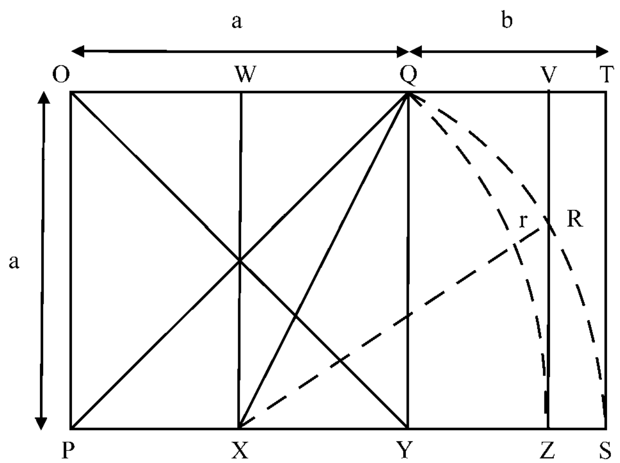

Fig. 6. Geometric construction of dynamic rectangles.

To generate dynamic rectangles that embed a series of roots and establish a golden ratio, we consider the geometric construction as shown in figure 6. If there is a square OPQY, and X is a midpoint on the side PY of the square OPQY, and we join X with Q and generate an arc with a radius *R* by rotating XQ clockwise, it intersects with the line extended along PXY in point S. If we join P with Q and generate an arc with a radius *r* by rotating PQ clockwise, it intersects with line PS in point Z. The generated shape OPST is a golden rectangle, and this reflects an annotation via the equation: $\frac{a+b}{a} = \frac{a}{b} \cong \varphi = \frac{1 + \sqrt{5}}{2} \cong 1.618\ldots$ Based on this, and by returning to the geometric construction of figure 5 earlier (regarding the arithmetic, geometric, and harmonic means), we refer to figure 7 in which the lines EC and CB act as the sides of the golden rectangle XBCE.

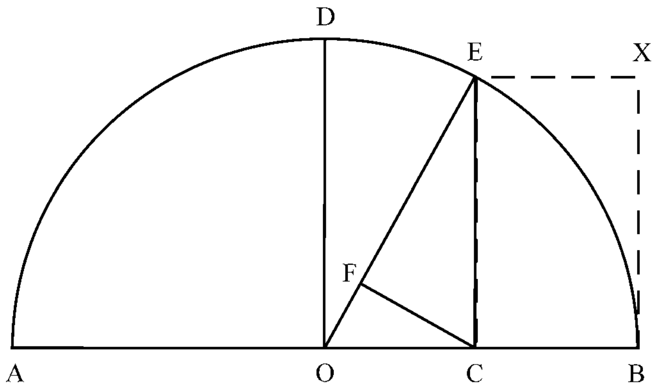

Fig. 7. Geometric derivation of the golden rectangle.

Moreover, and in order to generate a sequence of dynamic rectangles that are compressed, and that can act as proportional metrics for designing elevations, such as the proportional setting of the height of the various features of a façade, we can make the following geometric construction as shown in figure 8.

Let there be a square OPQR, and let there be an arc with a radius equal to OQ and that ends clockwise in point R. Let the line QP intersect with that arc at point *a*, then extend a line through *a* that is parallel to OP, so it intersects with PR in point S. We obtain in this the rectangle with sides QR and SR. Then let the line QS intersect with the arc at point *b*, and extend a line through *b* horizontally to intersect with PR in T, wherein we obtain a rectangle with sides QT and TR. Then let line QT intersect with the arc in *c*, and extend a line horizontally through *c* that intersects with PR in V, then we obtain a rectangle with sides QR and VR, and so forth. When unpacked, these sets of rectangles can give the shape of a sequence that is similar to what is shown in figure 9; namely, if the side of the original square has the value unit 1, then the sequence of rectangles

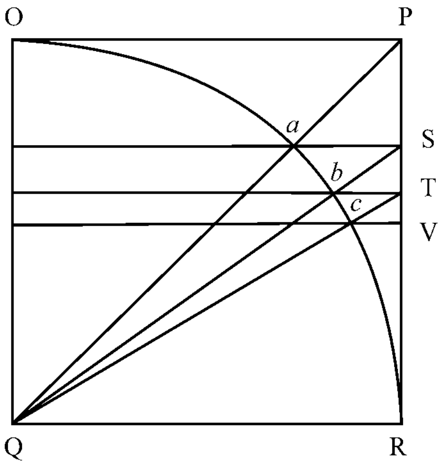

Fig. 8. Compressed dynamic rectangles.

will have their longer sides at the respective ascending values of: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, etc.

Using this accumulated geometric knowledge in that historical milieu, we analyze the horseshoe arch in the Hall of Richness according to what is shown in figure 10, which also discloses a relationship of rectangles that rest on an original square with its side having the value of a unit 1, and from which two rectangles are extracted with their longer sides having the respective values of $\sqrt{2}$, $\sqrt{3}$, and as shown on the right side of figure 10. The square with unit 1 corresponds with the height of the columns supporting the horseshoe arch and the distance between their axes as they also delimit the start of the circumference of the arch curvature, while

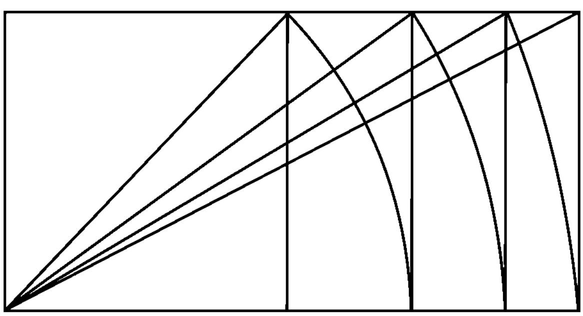

Fig. 9. Sequence of dynamic rectangles.

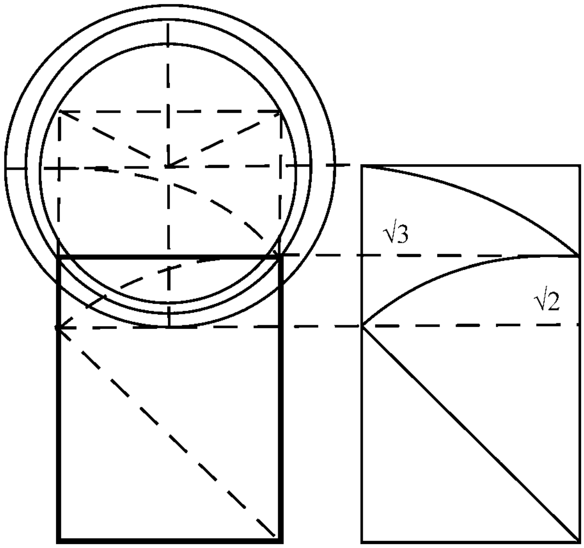

Fig. 10. Proportions of the horseshoe arch.

the rectangle with its longer side as $\sqrt{2}$ sets the point of convergence for the stucco decoration on the arch, and the rectangle with its longer side as $\sqrt{3}$ relates to the level of the main center of the concentric circles forming the upper parts of the arch.

Mathematized Paradigm

Based on the analysis set above, the invisible geometry that structures the formal properties of the visible architectural edifice reveals that a complex body of mathematical knowledge guided the processes of design. The elevational façade is interiorized within the opening of the courtyard, or in appearances within the inner spaces of the edifice. This is based on an interplay of the visible and invisible in a tapestry of architectural plan views and proportional elevations that guide the spatial ordering of the building and of its anthropometric embodiment. Such conditions in perception and architectural imagining also point to the unveiling of the seeming absence of the invisible behind the appearance of what is visible. This is a mode of revealing the hidden from behind the apparent (*kashf al-bāṭin al-maḥjūb bi-l-ẓāhir*), which is akin to revealing truth in the mode of ἀλήθεια (*alḗtheia*). Such spatial and optical aspects have a material determination to the statics and mechanics of the construction techniques, and the distribution of load-bearing structural engineering elements. Moreover, they embody environmental attributes in modulating the flow of heat, fresh air, cooling, and the gradational handling of the intensities of natural light. They also carry religious, social, and cultural attributes in the hierarchy of spatial gradations that shift from the urban and public domains to the inner private domains of architectural sanctuaries. Add to this the decorative ornamental architectonic features that generate stylistic, aesthetic, visual, and plastic art perceptual embodied experiences. In all of this, the uniform symmetry is the parity of the parts of the whole to one another, whereby each fragment corresponds to its opposite portion, as in the human figure. The arms, feet, hands, and fingers, are symmetrical and proportional qualities in relation to one another, and their anthropometric attributes apportion the constituting components of the architectural design. The scale is modular in the manner the human proportions guide the establishment of architectural orders. What is built is measured via an embodied experience in visual perception and its underlying optics (the science that develops into unparalleled levels of precision and impact via the mathematized experimental physics in the *Kitāb al-manāẓir* (Book of optics)[14] of Ibn al-Haytham (Latinized as Alhazen; d. ca. 431/1040), a contemporary of the Ikhwān al-Ṣafāʾ in Basra. This resonates with what was noted in Book I, chapter 2 of *De architectura libri decem* (The Ten Books on Architecture)[15] of Marcus Vitruvius Pollio, who notes that architecture depends on order *qua* τάξις (*táxis*) and disposition *qua* διάθεσις (*diáthesis*), while relying in this on proportion *qua* εὐρυθμία (*erythmía*), which in turn generates symmetry, decorum, and distribution through a stewardship of economy *qua* οἰκονομία (*oikonomía*). In this context, the sense of proportion points

14. Ibn al-Haytham, *Kitāb al-manāẓir* I-III (direct vision) (Kuwait 1983); Ibn al-Haytham, *Kitāb al-manāẓir* IV-V (catoptrics) (Kuwait 2002); N. El-Bizri, «Alhazen's Theory of Vision», in *Micrologus* 29 (2021), 21-33; N. El-Bizri, «A Philosophical Perspective on Alhazen's *Optics*», in *Arabic Sciences and Philosophy* 15 (2005), 189-218.

15. Vitruvius, *Ten Books on Architecture*, ed. I. Rowland, T. N. Howe (Cambridge 1999).

to a commodious harmony between the several components of a building and shows how these constitute the whole in an ordered design. This points to the generation of geometrical forms via arithmetical ratios in harmonious rhythmic modular orders (such as the thickness and height of the columns, the capital with the abacus cymatium, echinus, annulets, necking, and architrave)[16].

The mathematical analysis that underpins the design of architecture can also be linked to a symbolic order that is akin to the accounts of the Ikhwān al-Ṣafā' regarding the proportional ratios (*nisab*) and their applications in the *quadrivium* of arithmetic, geometry, astronomy, and music. This has bearing on studying the structure of the human body via a Neo-Pythagorean take on the microcosm/macrocosm analogy (respectively, *Epistle* 26: *al-insān ʿālam ṣaghīr*, and *Epistle* 34: *al-ʿālam insān kabīr*)[17]. This cosmological doctrine correlates, approximately, with the symmetries of the Platonic geometry of each regular polyhedron, which is isomorphic with one of the four elements (*arkān*), to determine the manifestation of corporeal matter, along with their common half triangular constitutive congruent edges that make up their surfaces (the four-faced sharp *tetrahedron*, for the hot and dry fire; the six-faced steady cube, for the cold and dry earth; the eight-faced blunt *octahedron*, for the hot and moist air; the twenty-faced smooth *icosahedron*, for the cold and moist water). These resonate with the Hippocratic-Galenic psychosomatic proportional balances of the four humors (*ruṭūbāt*) and their associated temperaments, as affected by the four elements and seasons: *sanguine* (*damawī*; *sanguin*), as the hot and damp *blood*, with the corresponding air and spring, and the temperament of sociability and hysteria; *phlegmatic* (*balghamī*; *flegmat*), as the cold and damp *phlegm*, with the correlative water and winter, and the supportive and schizoid temper; *melancholic* (*sawdāwī*; *melanc*), as the cold and

16. Such orders also became vital for the Italian Renaissance, in their deliberations over proportionality in the architecture of classical Doric, Ionic, and Corinthian styles, as noted, for instance, in Andrea Palladio's *Quattro libri dell'architettura* (Four books of architecture), Sebastiano Serlio's *Sette libri dell'architettura* (Seven books of architecture), and Giacomo Barozzi da Vignola's *Regola delli cinque ordini d'architettura* (Canon of the five orders of architecture).

17. I studied this elsewhere in N. El-Bizri, «Microcosm and Macrocosm: A Tentative Encounter between Graeco-Arabic Philosophy and Phenomenology», in *Islamic Philosophy and Occidental Phenomenology on the Perennial Issue of the Microcosm and Macrocosm*, ed. A.-T. Tymieniecka (Dordrecht 2006), 3-23.

dry *black bile*, which corresponds with earth and autumn, and the dispositions of reclusiveness and depression; *choleric* (*ghaḍabī*; *coleric*), as the hot and dry *yellow bile*, which correlates with fire and summer, and the irritability as well as dominance in mood[18].

On the whole, mathematics was the groundwork for τέχνη (*tékhnē*) as a technical craft, ποίησις (*poíēsis*) as a mode of production, ἐπιστήμη (*epistḗmē*) in theoretical erudition, and φρόνησις (*phrónēsis*) in practical applications. Mathematics also becomes a basis for ontological reflection in the way it aids the exercise of reasoning and imagining in unveiling the reality behind appearance, while also offering a paradigm for epistemology in terms of the theoretical and practical knowledge being mathematized. This ultimately enshrines the mathematical order of architecture via a Neo-Pythagorean penchant in thinking that resonates with the Vitruvian perspective on architecture, albeit by judiciously rooting it in the Islamicate milieu, with its building technicity and artistic craft.

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18. J. Jouanna, «The Legacy of the Hippocratic Treatise *The Nature of Man*: The Theory of the Four Humours», in *Greek Medicine from Hippocrates to Galen*, ed. Ph. van der Eijk, trans. N. Allies (Leiden 2012), 335-60.

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ABSTRACT

Nader El-Bizri, *The Mathematical Orders of Architecture: Seeing Madīnat al-Zahrā' from the Perspective of the Ikhwān al-Ṣafā'*

This article offers a mathematical analysis of some of the architectural features of the Salón Rico (Hall of Richness) of Madīnat al-Zahrā' (Radiant City in the outskirts of Córdoba), with a specific focus on the design and proportionality of the horseshoe arch. This line of inquiry is guided by the mathematical knowledge of that epoch in the Islamicate milieu of the fourth/tenth century, particularly as embodied in the studies on arithmetic, geometry, and proportional ratios in the *Rasā'il Ikhwān al-Ṣafā'* (Epistles of the Brethren of Purity); it also considers the construction techniques in that context and its associated arts and crafts.

Nader El-Bizri
University of Cambridge
elbizrinader@gmail.com